

On Ricci identities for submanifolds in the 2-osculator bundle

Oana Alexandru
"Transilvania" University of Brasov

Abstract

It is the purpose of the present paper to outline an introduction in theory of embeddings in the 2-osculator bundle. First, we recall the notion of 2-osculator bundle ([1],[2]) and the notion of submanifolds in the 2-osculator bundle. A moving frame is constructed. The induced connections and the relative covariant derivation are discussed in third and fourth sections. The Ricci identities for the deflection tensor are present in the next section.

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Introduction

Generally, the geometries of higher order defined as the study of the category of bundles of jet $(J_0^k M, \pi^k, M)$ is based on a direct approach of the properties of objects and morphisms in this category, without local coordinates.

But, many mathematical models from Lagrangian Mechanics, Theoretical Physics and Variational Calculus used multivariate Lagrangians of higher order acceleration.

From here one can see the reason of construction of the geometry of the total space of the bundle of higher accelerations (or the osculator bundle of higher order) in local coordinates.

As far we know the theory of Finsler submanifolds is far from being settled. In [8] and [9] R. Miron and M. Anastasiei give the theory of subspaces in generalized Lagrange spaces. Also, in [4] R. Miron presented the theory of subspaces in higher order Lagrange spaces. This article is draw upon the original construction of the higher order geometry given by R. Miron and Gh. Atanasiu ([4], [5],[6],[7]) and new aspects give by Gh. Atanasiu in [1].

If M is an immersed manifold in manifold M , a non linear connection on $Osc^2 M$ induce a nonlinear connection \tilde{N} on $Osc^2 \tilde{M}$. We take the metrical

N-linear connection D on the manifold $Osc^2 M$. This allows obtain of the induced tangent and normal connections and introduction of the relative covariant derivation in the algebra of d-tensor fields ([?]). If in [4] R. Miron use the canonical metrical N-linear connection of the space $L^{(2)n}$ having the coefficients $\left(F_{jk}^i, C_{1jk}^i, C_{2jk}^i\right)$, in this article we take the canonical metrical N-linear connection of the manifold $Osc^2 M$ having the coefficients $\left(L_{(i0)bc}^a, C_{(i1)bc}^a, C_{(i2)bc}^a\right)$, $(i = 0, 1, 2)$.

In this paper we get the Ricci identities for the Liouville d-vector fields $z^{(1)\alpha}$ and $z^{(2)\alpha}$ (Theorem 5.2). The same problem was solved by prof. Atanasiu Gh. in [1] for the Liouville d-vector fields $z^{(1)a}$ and $z^{(2)a}$.

1 The 2-osculator bundle $(Osc^2 M, \pi^2, M)$

Let M be a real differentiable manifold of dimension n . A point of M will be denoted by x and its local coordinate system by (U, φ) , $\varphi(x) = (x^a)$. The indices a, b, \dots run over set $\{1, 2, \dots, n\}$ and Einstein convention of summarizing is adopted all over this work.

Let us consider two curves $\rho, \sigma : I \rightarrow M$, having images in a domain of local chart $U \subset M$. We say that ρ and σ have a "contact of order 2" in a point $x_0 \in U$ if: $\rho(0) = \sigma(0) = x_0$, $(0 \in I)$, and for any function $f \in \mathcal{F}(U)$:

$$\frac{d^\beta}{dt^\beta} (f \circ \rho)(t) \big|_{t=0} = \frac{d^\beta}{dt^\beta} (f \circ \sigma)(t) \big|_{t=0}, (\beta = 1, 2) \quad (1.1)$$

The relation "contact of order 2" is an equivalence on the set of smooth curves in M , which pass through the point x_0 . Let $[\rho]_{x_0}$ be a class of equivalence. It will be called a "2-osculator space" in a point $x_0 \in M$. The set of 2-osculator spaces in the point $x_0 \in M$ will be denoted by $Osc_{x_0}^2 M$, and we put

$$Osc^2 M = \bigcup_{x_0 \in M} Osc_{x_0}^2 M$$

One considers the mapping $\pi^2 : Osc^2 M \rightarrow M$ define by $\pi^2([\rho]_{x_0}) = x_0$. Obviously, π^2 is a surjection.

The set $Osc^2 M$ is endowed with a natural differentiable structure, induced by that of the manifold M , so that π^2 is a differentiable mapping. It will be described below.

The curve $\rho : I \rightarrow M$ ($\text{Im } \rho \subset U$) is analytically represented in the local chart (U, φ) by $x_0 = x_0^a (= x^a(0))$. Taking the function f from 1.1, successively equal to the coordinate functions x^a , then a representative of the class $[\rho]_{x_0}$ is given by

$$x^{*a}(t) = x^a(0) + t \frac{dx^a}{dt}(0) + \frac{1}{2} t^2 \frac{d^2 x^a}{dt^2}(0), \quad t \in (-\varepsilon, \varepsilon) \subset I.$$

The previous polynomials are determined by the coefficients

$$x_0^a = x^a(0), y^{(1)a} = \frac{dx^a}{dt}(0), y^{(2)a} = \frac{1}{2} \frac{d^2x^a}{dt^2}(0) \quad (1.2)$$

Hence, the pair $\left((\pi^2)^{-1}(U), \Phi\right)$, with $\Phi([\rho]_{x_0}) = (x_0^a, y^{(1)a}, y^{(2)a}) \in R^{3n}$, $\forall [\rho]_{x_0} \in (\pi^2)^{-1}(U)$ is a local chart on Osc^2M . Thus a differentiable atlas \mathcal{A}_M of the differentiable structure on the manifold M determines a differentiable atlas A_{Osc^2M} on Osc^2M and therefore the triple (Osc^2M, π^2, M) is a differentiable bundle. We will identify the 2-oscillator bundle (Osc^2M, π^2, M) with 2-tangent bundle (T^2M, π^2, M) .

By 1.2, a transformation of local coordinates $(x^a, y^{(1)a}, y^{(2)a}) \rightarrow (\tilde{x}^a, \tilde{y}^{(1)a}, \tilde{y}^{(2)a})$ on the manifold Osc^2M is given by

$$\begin{cases} \tilde{x}^a = \tilde{x}^a(x^1, \dots, x^n), \det\left(\frac{\partial \tilde{x}^a}{\partial x^b}\right) \neq 0 \\ \tilde{y}^{(1)a} = \frac{\partial \tilde{x}^a}{\partial x^b} y^{(1)b} \\ 2\tilde{y}^{(2)a} = \frac{\partial \tilde{y}^{(1)a}}{\partial x^b} y^{(1)b} + 2 \frac{\partial \tilde{y}^{(1)a}}{\partial y^{(1)b}} y^{(2)b} \end{cases} \quad (1.3)$$

One can see that Osc^2M is of dimension $3n$.

Let us consider the **2-tangent structure** \mathbb{J} on Osc^2M

$$\mathbb{J}\left(\frac{\partial}{\partial x^a}\right) = \frac{\partial}{\partial y^{(1)a}}, \quad \mathbb{J}\left(\frac{\partial}{\partial y^{(1)a}}\right) = \frac{\partial}{\partial y^{(2)a}}, \quad \mathbb{J}\left(\frac{\partial}{\partial y^{(2)a}}\right) = 0$$

where $\left(\frac{\partial}{\partial x^a} \big|_u, \frac{\partial}{\partial y^{(1)a}} \big|_u, \frac{\partial}{\partial y^{(2)a}} \big|_u\right)$ is the natural basis of tangent space

$T_u Osc^2M, u \in Osc^2M$. If N is a nonlinear connection on Osc^2M , then $N_0 = N$, $\mathbb{J}(N_0) = N_1$ are two distributions geometrically defined on Osc^2M , all of dimension n . Let us consider the distributions V_2 on Osc^2M locally generated by the vector fields $\left\{\frac{\partial}{\partial y^{(2)a}}\right\}$. Consequently, the tangent bundle to Osc^2M at the point $u \in Osc^2M$ is given by a direct sum of the vector space:

$$T_u Osc^2M = N_0(u) \oplus N_1(u) \oplus V_2(u), \forall u \in Osc^2M. \quad (1.5)$$

We consider $\left\{\frac{\delta}{\delta x^a}, \frac{\delta}{\delta y^{(1)a}}, \frac{\delta}{\delta y^{(2)a}}\right\}$ an adapted basis to the decomposition 1.5 and its dual basis denoted by $(dx^a, \delta y^{(1)a}, \delta y^{(2)a})$, where

$$\begin{cases} dx^a = & dx^a \\ \delta y^{(1)a} = & dy^{(1)a} + M_{(1)b}^a dx^b \\ \delta y^{(2)a} = & dy^{(2)a} + M_{(1)b}^a \delta y^b + M_{(2)b}^a \delta y^{(2)b} \end{cases} \quad (1.7)$$

Definition 1.1 A linear connection D on $Osc^2 M$ is called **N-linear connection** if it preserves by parallelism the horizontal and vertical distribution N_0, N_1 and V_2 on $Osc^2 M$.

Any N-linear connection D can be represented by an unique system of functions $D\Gamma(N) = \left(L_{(i0)bd}^a, C_{(i1)bd}^a, C_{(i2)bd}^a \right), (i = 0, 1, 2)$. These functions are called **the coefficients** of the N-linear connection D .

If on the manifold $Osc^2 M$ is given a N-linear connection D then there exists a h_i -, v_{1i} - and v_{2i} -**covariant derivatives** in local adapted basis $(i = 0, 1, 2)$.

Any d-tensor T , of type (r, s) can be represented in the adapted basis and its dual basis in the form

$$T = T_{b_1 \dots b_s}^{a_1 \dots a_r} \delta_{a_1} \otimes \dots \otimes \dot{\partial}_{2a_r} \otimes dx^{b_1} \otimes \dots \otimes \delta y^{(2)b_s}.$$

and we have

$$\begin{aligned} T_{b_1 \dots b_s}^{a_1 \dots a_r} \Big|_{id} &= \delta_a T_{b_1 \dots b_s}^{a_1 \dots a_r} + L_{(i0)cd}^{a_1} T_{b_1 \dots b_s}^{ca_2 \dots a_r} + \dots + \\ &+ L_{(i0)cd}^{a_r} T_{b_1 \dots b_s}^{a_1 \dots a_{r-1}c} - L_{(i0)b_1d}^c T_{cb_2 \dots b_s}^{a_1 \dots a_r} - \dots - L_{(i0)b_sd}^c T_{cb_2 \dots b_{s-1}c}^{a_1 \dots a_r} \\ T_{b_1 \dots b_s}^{a_1 \dots a_r} \Big|_{id}^{(1)} &= \delta_{1a} T_{b_1 \dots b_s}^{a_1 \dots a_r} + C_{(i1)cd}^{a_1} T_{b_1 \dots b_s}^{ca_2 \dots a_r} + \dots + \\ &+ C_{(i1)cd}^{a_r} T_{b_1 \dots b_s}^{a_1 \dots a_{r-1}c} - C_{(i1)b_1d}^c T_{cb_2 \dots b_s}^{a_1 \dots a_r} - \dots - C_{(i1)b_sd}^c T_{cb_2 \dots b_{s-1}c}^{a_1 \dots a_r} \\ T_{b_1 \dots b_s}^{a_1 \dots a_r} \Big|_{id}^{(2)} &= \delta_{2a} T_{b_1 \dots b_s}^{a_1 \dots a_r} + C_{(i2)cd}^{a_1} T_{b_1 \dots b_s}^{ca_2 \dots a_r} + \dots + \\ &+ C_{(i2)cd}^{a_r} T_{b_1 \dots b_s}^{a_1 \dots a_{r-1}c} - C_{(i2)b_1d}^c T_{cb_2 \dots b_s}^{a_1 \dots a_r} - \dots - C_{(i2)b_sd}^c T_{cb_2 \dots b_{s-1}c}^{a_1 \dots a_r} \\ &\left(\delta_{1a} = \frac{\delta}{\delta y^{(1)a}}, \delta_{2a} = \frac{\delta}{\delta y^{(2)a}}; i = 0, 1, 2 \right) \end{aligned}$$

The operators " $\Big|_{id}$ ", " $\Big|_{id}^{(1)}$ " and " $\Big|_{id}^{(2)}$ " are called the h_i -, v_{1i} - and v_{2i} -**covariant derivatives** with respect to $D\Gamma(N)$.

Definition 1.2 A N-linear connection D on $Osc^2 M$ endowed with a structure metric \mathbb{G} is said to be a **metric N-linear connection** if $D_X \mathbb{G} = 0$ for every $X \in X(Osc^2 M)$.

2 Submanifolds in the 2-osculator bundle

Let M be a C^∞ real, n -dimensional manifold and \check{M} be a real, m -dimensional manifold, immersed in M through the immersion $i : \check{M} \rightarrow M$. Locally, i can be given in the form

$$x^a = x^a(u^1, \dots, u^m), \quad \text{rank} \left\| \frac{\partial x^a}{\partial u^\alpha} \right\| = m \quad (2.1)$$

The indices a, b, c, \dots run over the set $\{1, \dots, n\}$ and $\alpha, \beta, \gamma, \dots$ run on the set $\{1, \dots, m\}$. We assume $1 < m < n$. If i is an embedding, then we identify \check{M} to $i(\check{M})$ and say that \check{M} is a submanifold of the manifold M . Therefore 2.1 will be called the parametric equations of the submanifold \check{M} in the manifold M .

The embedding $i : \check{M} \rightarrow M$ determines an immersion $Osc^2 i : Osc^2 \check{M} \rightarrow Osc^2 M$, defined by the covariant functor $Osc^2 : Man \rightarrow Man$. ([4])

The mapping $Osc^2 i : Osc^2 \check{M} \rightarrow Osc^2 M$ has the parametric equations:

$$\begin{cases} x^a = x^a(u^1, \dots, u^m), \text{rank} \left\| \frac{\partial x^a}{\partial u^\alpha} \right\| = m \\ y^{(1)a} = \frac{\partial x^a}{\partial u^\alpha} v^{(1)\alpha} \\ 2y^{(2)a} = \frac{\partial y^{(1)a}}{\partial u^\alpha} v^{(1)\alpha} + 2 \frac{\partial y^{(1)a}}{\partial v^{(1)\alpha}} v^{(2)\alpha} \end{cases} \quad (2.2)$$

where

$$\begin{cases} \frac{\partial x^a}{\partial u^\alpha} = \frac{\partial y^{(1)a}}{\partial v^{(1)\alpha}} = \frac{\partial y^{(2)a}}{\partial v^{(2)\alpha}} \\ \frac{\partial y^{(1)a}}{\partial u^\alpha} = \frac{\partial y^{(2)a}}{\partial v^{(1)\alpha}} \end{cases} \quad (2.3)$$

The Jacobian matrix of 2.2 is $J(Osc^2 i)$ and it has the rank equal to $3m$. So, $Osc^2 i$ is an immersion. The differential i_* of the mapping $Osc^2 i : Osc^2 \check{M} \rightarrow Osc^2 M$ leads to the relation between the natural basis of the modules $\mathcal{X}(Osc^2 \check{M})$ and $\mathcal{X}(Osc^2 M)$ given by

$$i_* \left\| \frac{\partial}{\partial u^\alpha} \frac{\partial}{\partial v^{(1)\alpha}} \dots \frac{\partial}{\partial v^{(2)\alpha}} \right\| = \left\| \frac{\partial}{\partial x^a} \frac{\partial}{\partial y^{(1)a}} \dots \frac{\partial}{\partial y^{(2)a}} \right\| J(Osc^2 i).$$

This differential i_* maps the cotangent space $T^*(Osc^2 M)$ in a point of $Osc^2 M$,

into the cotangent space $T^*(Osc^2\check{M})$ in a point of $Osc^2\check{M}$ by the rule:

$$\begin{aligned} dx^a &= \frac{\partial x^a}{\partial u^\alpha} du^\alpha \\ dy^{(1)a} &= \frac{\partial y^{(1)a}}{\partial u^\alpha} du^\alpha + \frac{\partial y^{(1)a}}{\partial v^{(1)\alpha}} dv^{(1)\alpha} \\ dy^{(2)a} &= \frac{\partial y^{(2)a}}{\partial u^\alpha} du^\alpha + \frac{\partial y^{(2)a}}{\partial v^{(1)\alpha}} dv^{(1)\alpha} + \frac{\partial y^{(2)a}}{\partial v^{(2)\alpha}} dv^{(2)\alpha} \end{aligned} \quad (2.4)$$

We used the previous theory for study the induced geometrical object fields from Osc^2M to $Osc^2\check{M}$.

Let us consider

$$B_\alpha^a = \frac{\partial x^a}{\partial u^\alpha}. \quad (2.5)$$

We take g_{ab} a nondegenerate metric on the manifold M and $\mathbb{G} = g_{ab}dx^a \otimes dx^b + g_{ab}\delta y^{(1)a} \otimes \delta y^{(1)a} + g_{ab}\delta y^{(2)a} \otimes \delta y^{(2)a}$ the Sasaki prolongation of the metric g along $Osc^2\check{M}$.

Thus, $\{B_1^a, B_2^a, \dots, B_m^a\}$ are m-linear independent d-vector fields on $Osc^2\check{M}$. Also, $\{B_\alpha^1, B_\alpha^2, \dots, B_\alpha^n\}$ are d-covector fields, with respect to the next transformations of coordinates

$$\left\{ \begin{aligned} \bar{u}^\alpha &= \bar{u}^\alpha(u^1, \dots, u^m), \text{rank} \left\| \frac{\partial \bar{u}^\alpha}{\partial u^\beta} \right\| = m \\ \bar{v}^{(1)\alpha} &= \frac{\partial \bar{u}^\alpha}{\partial u^\beta} v^{(1)\beta} \\ 2\bar{v}^{(2)\alpha} &= \frac{\partial \bar{v}^{(1)\alpha}}{\partial u^\beta} v^{(1)\beta} + 2 \frac{\partial \bar{v}^{(1)\alpha}}{\partial v^{(1)\beta}} v^{(2)\beta} \end{aligned} \right. \quad (2.6)$$

Of course, d-vector fields $\{B_1^a, \dots, B_m^a\}$ are tangent to the submanifold \check{M} .

We say that a d-vector field $\xi^a(x, y^{(1)}, y^{(2)})$ is **normal** to $Osc^2\check{M}$ if, on $\check{\pi}^{-1}(\check{U}) \subset Osc^2\check{M}$, we have

$$\begin{aligned} g_{ab}(x(u), y^{(1)}(u, v^{(1)}, v^{(2)}), y^{(2)}(u, v^{(1)}, v^{(2)})) B_\alpha^a(u) \cdot \\ \cdot \xi^b(x(u), y^{(1)}(u, v^{(1)}, v^{(2)}), y^{(2)}(u, v^{(1)}, v^{(2)})) = 0 \end{aligned}$$

Consequently, on $\check{\pi}^{-1}(\check{U}) \subset Osc^2\check{M}$ there exist $n - m$ unit vector fields $B_{\bar{\alpha}}^a$, ($\bar{\alpha} = 1, \dots, n - m$) normal along $Osc^2\check{M}$, and to each other:

$$g_{ab} B_{\bar{\alpha}}^a B_{\bar{\beta}}^b = 0, \quad g_{ab} B_{\bar{\alpha}}^a B_{\bar{\beta}}^b = \delta_{\bar{\alpha}\bar{\beta}}, \quad (\bar{\alpha}, \bar{\beta} = 1, \dots, n - m) \quad (2.7)$$

The system of d-vectors $B_{\bar{\alpha}}^a$ ($\bar{\alpha} = 1, \dots, n - m$) is determined up to orthogonal transformations of the form

$$B_{\bar{\alpha}'}^a = A_{\bar{\alpha}'}^{\bar{\beta}} B_{\bar{\beta}}^a, \quad \|A_{\bar{\alpha}'}^{\bar{\beta}}\| \in \mathcal{O}(n - m), \quad (2.8)$$

where $\bar{\alpha}, \bar{\beta}, \dots$ run over the set $(1, 2, \dots, n - m)$.

We say that the system of d-vectors $\{B_\alpha^a, B_{\bar{\alpha}}^a\}$ determines a frame in $Osc^2 M$ along to $Osc^2 \check{M}$.

Its dual frame will be denoted by $\{B_\alpha^a(u, v^{(1)}, \dots, v^{(2)}), B_{\bar{\alpha}}^a(u, v^{(1)}, \dots, v^{(2)})\}$. This is also defined on an open set $\tilde{\pi}^{-1}(\check{U}) \subset Osc^2 \check{M}$, \check{U} being a domain of a local chart on the submanifold \check{M} .

The conditions of duality are given by:

$$B_\beta^a B_\alpha^a = \delta_\beta^\alpha, \quad B_\beta^a B_{\bar{\alpha}}^a = 0, \quad B_\alpha^a B_{\bar{\beta}}^a = 0, \quad B_{\bar{\alpha}}^a B_{\bar{\beta}}^a = \delta_{\bar{\beta}}^{\bar{\alpha}} \quad (2.9)$$

$$B_\alpha^a B_b^a + B_{\bar{\alpha}}^a B_{\bar{b}}^a = \delta_b^a \quad (2.9')$$

Using 2.7, we deduce:

$$g_{\alpha\beta} B_\alpha^a = g_{ab} B_\beta^a, \quad \delta_{\bar{\alpha}\bar{\beta}} B_{\bar{b}}^{\bar{\beta}} = g_{ab} B_{\bar{\alpha}}^a. \quad (2.10)$$

So, we can look to the set

$$\mathcal{R} = \left\{ \left(u, v^{(1)}, v^{(2)} \right); B_\alpha^a(u), B_{\bar{\alpha}}^a \left(u, v^{(1)}, v^{(2)} \right) \right\}$$

$(u, v^{(1)}, v^{(2)}) \in \tilde{\pi}^{-1}(\check{U})$ as a moving frame. Now, we shall represent in \mathcal{R} the d-tensor fields from the space $Osc^2 M$, restricted to the open set $\tilde{\pi}^{-1}(\check{U})$.

3 Induced nonlinear connections

Now, let us consider the canonical nonlinear connection N on the $Osc^2 M$. Then its dual coefficients $M_{(1)b}^a, M_{(2)b}^a$ depends only by the metric g . We will prove that

the restriction of the of the nonlinear connection N to $Osc^2 \check{M}$ uniquely determines an induced nonlinear connection \check{N} on $Osc^2 \check{M}$. Of course, \check{N} is well determined by means of its dual coefficients $\left(\check{M}_{(1)\beta}^\alpha, \check{M}_{(2)\beta}^\alpha \right)$ or by means of its adapted cobasis $(du^\alpha, \delta v^{(1)\alpha}, \delta v^{(2)\alpha})$.

Definition 3.1 A non-linear connection \check{N} on $Osc^2 \check{M}$ is called **induced** by the nonlinear connection N if we have

$$\delta v^{(1)\alpha} = B_\alpha^a \delta y^{(1)a}, \quad \delta v^{(2)\alpha} = B_\alpha^a \delta y^{(2)a} \quad (3.1)$$

Proposition 3.1 The dual coefficients of the non-linear connection \check{N} are

$$\begin{aligned} \check{M}_{(1)\beta}^\alpha &= B_\alpha^a \left(B_{0\beta}^a + M_{(1)b}^a B_\beta^b \right) \\ \check{M}_{(2)\beta}^\alpha &= B_\alpha^a \left(\frac{1}{2} \frac{\partial B_{\delta\gamma}^a}{\partial u^\beta} v^{(1)\delta} v^{(1)\gamma} + B_{\delta\beta}^a v^{(2)\delta} + M_{(1)b}^a B_{0\beta}^b + M_{(2)b}^a B_\beta^b \right) \end{aligned} \quad (3.2)$$

where $M_{(1)b}^a, M_{(2)b}^a$ are the dual coefficients of the non-linear connection N .

Theorem 3.1 *The cobasis $(dx^a, \delta y^{(1)a}, \delta y^{(2)a})$ restricted to $Osc^2 \check{M}$ is uniquely represented in the moving frame \mathcal{R} in the following form:*

$$\begin{cases} dx^a = B_\beta^a du^\beta \\ \delta y^{(1)a} = B_\alpha^a \delta v^{(1)\alpha} + B_{\bar{\alpha}(1)}^a K_{\bar{\beta}}^{\bar{\alpha}} du^\beta \\ \delta y^{(2)a} = B_\alpha^a \delta v^{(2)\alpha} + B_{\bar{\beta}(1)}^a K_{\bar{\alpha}}^{\bar{\beta}} \delta v^{(1)\alpha} + B_{\bar{\beta}(2)}^a K_{\bar{\alpha}}^{\bar{\beta}} du^\alpha \end{cases} \quad (3.5)$$

where

$$\begin{aligned} K_{(1)\bar{\beta}}^{\bar{\alpha}} &= B_a^{\bar{\alpha}} \left(B_{0\beta}^a + M_{(1)b}^a B_\beta^b \right) \\ K_{(2)\bar{\beta}}^{\bar{\alpha}} &= B_a^{\bar{\alpha}} \left(\frac{1}{2} \frac{\partial B_{\delta\gamma}^a}{\partial u^\beta} v^{(1)\delta} v^{(1)\gamma} + B_{\delta\beta}^b v^{(2)\delta} + M_{(1)b}^a B_{0\beta}^b + M_{(2)b}^a B_\beta^b - \right. \\ &\quad \left. - B_f^{\bar{\alpha}} B_d^\gamma \left(B_\gamma^f + M_{(1)b}^f B_\gamma^b \right) \left(B_{0\beta}^d + M_{(1)g}^d B_\beta^g \right) \right) \end{aligned} \quad (3.6)$$

are mixed d-tensor fields.

Proof. The first relation is obviously. From 2.2 and 3.2 we obtain 3.5. ■

Generally, a set of functions $T_{j\dots\beta\dots\bar{\beta}}^{i\dots\alpha\dots\bar{\alpha}}(u, v^{(1)}, v^{(2)})$ which are d-tensors in the index i, j, \dots , and d-tensors in the index α, β, \dots and tensors with respect to the transformations 2.8 in the index $\bar{\alpha}, \bar{\beta}, \dots$ is called a *mixed* d-tensor field on $Osc^2 \check{M}$.

4 The relative covariant derivatives

We shall construct the operators $\nabla_{(i)}$ of relative (or mixed) covariant derivation in the algebra of mixed d-tensor fields. It is clear that $\nabla_{(i)}$ will be known if its action of functions and on the vector fields of the form

$$\begin{aligned} X^a(x(u), y^{(1)}(u, v^{(1)}), y^{(2)}(u, v^{(1)}, v^{(2)})) \\ X^\alpha(u, v^{(1)}, v^{(2)}), X^{\bar{\alpha}}(u, v^{(1)}, v^{(2)}) \end{aligned} \quad (4.1)$$

are known.

Definition 4.1 The **coupling** of the metrical d-connection D to the induced nonlinear connection \check{N} along $Osc^2 \check{M}$ is locally given by the set of its nine coefficients $\check{D}\Gamma(\check{N}) = \left(\check{L}_{(i0)}^a, \check{C}_{(i1)}^a, \check{C}_{(i2)}^a \right), (i = 0, 1, 2)$ where

$$\begin{aligned}
\check{L}_{(i0)}^a{}_{b\delta} &= L_{(i0)}^a{}_{bd} B_\delta^d + C_{(i1)}^a{}_{bd} B_\delta^d K_{(1)\delta}^{\bar{\delta}} + C_{(i2)}^a{}_{b\delta} B_\delta^d K_{(2)\delta}^{\bar{\delta}} \\
\check{C}_{(i1)}^a{}_{b\delta} &= C_{(i1)}^a{}_{bd} B_\delta^d + C_{(i2)}^a{}_{bd} B_\delta^d K_{(1)\delta}^{\bar{\delta}} \quad (i = 0, 1, 2) \\
\check{C}_{(i2)}^a{}_{b\delta} &= C_{(i2)}^a{}_{bd} B_\delta^d
\end{aligned} \tag{4.5}$$

We have the operators $\check{D}_{(i)}$ and $D_{(i)}$ ($i = 0, 1, 2$) with the property

$$\check{D}_{(i)} X^a = D_{(i)} X^a \text{ (modulo 3.5)} \tag{4.2}$$

where

$$D_{(i)} X^a = dX^a + X^b \omega_{(i)b}^a, \tag{4.3}$$

and

$$\check{D}_{(i)} X^a = dX^a + X^b \check{\omega}_{(i)b}^a \tag{4.4}$$

Here $\omega_{(i)b}^a$ and $\check{\omega}_{(i)b}^a$ are the 1-forms of the metrical d-connection D and the coupling \check{D} respectively.

Of course, we can write $\check{D}_{(i)} X^a$ in the form

$$\check{D}_{(i)} X^a = X^a \mid_{i\delta} du^\delta + X^a \mid_{i\delta}^{(1)} \delta v^{(1)\delta} + X^a \mid_{i\delta}^{(2)} \delta v^{(2)\delta}.$$

Definition 4.2 We call the **induced tangent connection** on $Sc^2 \check{M}$ by the metrical d-connection D the set of its nine coefficients $D^\top \Gamma(\check{N}) = \left(L_{(i0)}^\alpha{}_{\beta\delta}, C_{(i1)}^\alpha{}_{\beta\delta}, C_{(i2)}^\alpha{}_{\beta\delta} \right)$ ($i = 0, 1, 2$) where

$$\begin{aligned}
L_{(i0)}^\alpha{}_{\beta\delta} &= B_d^\alpha \left(B_{\beta\delta}^d + B_\beta^f \check{L}_{(i0)f\delta}^d \right) \\
C_{(i1)}^\alpha{}_{\beta\delta} &= B_d^\alpha B_\beta^f \check{C}_{(i1)f\delta}^d \quad (i = 0, 1, 2) \\
C_{(i2)}^\alpha{}_{\beta\delta} &= B_d^\alpha B_\beta^f \check{C}_{(i2)f\delta}^d
\end{aligned} \tag{4.9}$$

We have the operators $D_{(i)}^\top$ with the properties

$$D_{(i)}^\top X^\alpha = B_b^\alpha \check{D}_{(i)} X^b, \quad \text{for } X^a = B_\gamma^a X^\gamma \tag{4.6}$$

$$D_{(i)}^\top X^\alpha = dX^\alpha + X_{(i)}^\beta \omega_{\beta}^\alpha \quad (4.7)$$

where ω_{β}^α are the **connection 1-forms** of $D_{(i)}^\top$ ($i = 0, 1, 2$).

As in the case of \check{D} we may write

$$D_{(i)}^\top X^\alpha = X_{|i\delta}^\alpha du^\delta + X^\alpha \big|_{i\delta}^{(1)} \delta v^{(1)\delta} + X^\alpha \big|_{i\delta}^{(2)} \delta v^{(2)\delta}.$$

Definition 4.3 We call the **induced normal connection** on $Osc^2 \check{M}$ by the metrical d-connection D the set of its nine coefficients $D^\perp \Gamma(\check{N}) = \left(L_{(i0)\check{\beta}\delta}^{\check{\alpha}}, C_{(i1)\check{\beta}\delta}^{\check{\alpha}}, C_{(i2)\check{\beta}\delta}^{\check{\alpha}} \right)$ where

$$\begin{aligned} L_{(i0)\check{\beta}\delta}^{\check{\alpha}} &= B_d^{\check{\alpha}} \left(\frac{\delta B_{\check{\beta}}^d}{\delta u^\delta} + B_{\check{\beta}(i0)}^f \check{L}_{f\delta}^d \right) \\ C_{(i1)\check{\beta}\delta}^{\check{\alpha}} &= B_d^{\check{\alpha}} \left(\frac{\delta B_{\check{\beta}}^d}{\delta v^{(1)\delta}} + B_{\check{\beta}(i1)}^f \check{C}_{f\delta}^d \right) \quad (i = 0, 1, 2) \\ C_{(i2)\check{\beta}\delta}^{\check{\alpha}} &= B_d^{\check{\alpha}} \left(\frac{\partial B_{\check{\beta}}^d}{\partial v^{(2)\delta}} + B_{\check{\beta}(i2)}^f \check{C}_{f\delta}^d \right) \end{aligned} \quad (4.13)$$

As before, we have the operators $D_{(i)}^\perp$ with the properties

$$D_{(i)}^\perp X^{\check{\alpha}} = B_b^{\check{\alpha}} \check{D}_{(i)} X^b, \quad \text{for } X^a = B_{\check{\gamma}}^a X^{\check{\gamma}} \quad (4.10)$$

$$D_{(i)}^\perp X^{\check{\alpha}} = dX^{\check{\alpha}} + X_{(i)}^{\check{\beta}} \omega_{\check{\beta}}^{\check{\alpha}} \quad (4.11)$$

where $\omega_{\check{\beta}}^{\check{\alpha}}$ are the **connection 1-forms** of $D_{(i)}^\perp$ ($i = 0, 1, 2$).

We may set

$$D_{(i)}^\perp X^{\check{\alpha}} = X_{|i\delta}^{\check{\alpha}} du^\delta + X^{\check{\alpha}} \big|_{i\delta}^{(1)} \delta v^{(1)\delta} + X^{\check{\alpha}} \big|_{i\delta}^{(2)} \delta v^{(2)\delta}.$$

Now, we can define the relative (or mixed) covariant derivatives $\nabla_{(i)}$ enounced at the begining of this section.

Theorem 4.4 *A relative (mixed) covariant derivation in the algebra of mixed d-tensor fields is an operator $\nabla_{(i)}$ for which the following properties hold:*

$$\nabla_{(i)} f = df, \quad \forall f \in \mathcal{F}(Osc^2 \check{M})$$

$$\nabla_{(i)} X^a = \check{D}_{(i)} X^a, \quad \nabla_{(i)} X^\alpha = D_{(i)}^\top X^\alpha, \quad \nabla_{(i)} X^{\check{\alpha}} = D_{(i)}^\perp X^{\check{\alpha}} \quad (i = 0, 1, 2)$$

The connection 1-forms $\tilde{\omega}_{(i)b}^a, \omega_{(i)\beta}^\alpha, \omega_{(i)\bar{\beta}}^{\bar{\alpha}}$ will be called the **connection 1-forms** of $\nabla_{(i)}$.

The **Liouville vector fields** for submanifolds, introduced by professor Miron in [4], are

$$\begin{aligned}\mathbb{C}_1 &= v^{(1)\alpha} \frac{\partial}{\partial v^{(2)\alpha}} \\ \mathbb{C}_2 &= v^{(1)\alpha} \frac{\partial}{\partial v^{(1)\alpha}} + 2v^{(2)\alpha} \frac{\partial}{\partial v^{(2)\alpha}}.\end{aligned}$$

If we represent this vector fields in the adapted basis, we get

$$\mathbb{C}_1 = z^{(1)\alpha} \dot{\partial}_{2\alpha}, \mathbb{C}_2 = z^{(1)\alpha} \delta_{1\alpha} + 2z^{(2)\alpha} \dot{\partial}_{2\alpha}$$

where

$$z^{(1)\alpha} = v^{(1)\alpha}, z^{(2)\alpha} = v^{(2)\alpha} + \frac{1}{2} M_{(1)}^\alpha{}_\beta v^{(1)\beta}.$$

D -vector fields $z^{(1)\alpha}$ and $z^{(2)\alpha}$ are called the **Liouville d-vector fields** of submanifolds $Osc^2 M$.

The $(z^{(1)})$ - and $(z^{(2)})$ -**deflection** tensor fields are:

$$\begin{aligned}z^{(1)\alpha} |_{i\beta} &= D_i^\alpha{}_\beta, & z^{(1)\alpha} |_{i\beta} &= d_i^{(1)\alpha}{}_\beta, & z^{(1)\alpha} |_{i\beta} &= d_i^{(2)\alpha}{}_\beta, \\ z^{(2)\alpha} |_{i\beta} &= D_i^{(2)\alpha}{}_\beta, & z^{(2)\alpha} |_{i\beta} &= d_i^{(21)\alpha}{}_\beta, & z^{(2)\alpha} |_{i\beta} &= d_i^{(22)\alpha}{}_\beta.\end{aligned}\tag{5.1}$$

Proposition 4.1 The $(z^{(1)})$ -deflection fields have the expression:

$$\begin{aligned}D_i^{(1)\alpha} &= -N_1^\alpha{}_\beta + z^{(1)\gamma} L_{(i0)\gamma\beta}^\alpha, \\ d_i^{(11)\alpha}{}_\beta &= \delta_\beta^\alpha + z^{(1)\gamma} C_{(i1)\gamma\beta}^\alpha, \\ d_i^{(12)\alpha}{}_\beta &= z^{(1)\gamma} C_{(i2)\gamma\beta}^\alpha.\end{aligned}\tag{5.2}$$

Indeed, if we take

$$\begin{aligned}z^{(1)\alpha} |_{i\beta} &= \delta_\beta^\alpha z^{(1)\alpha} + z^{(1)\gamma} L_{(i0)\gamma\beta}^\alpha, \\ z^{(1)\alpha} |_{i\beta}^{(j)} &= \delta_{j\beta}^\alpha z^{(1)\alpha} + z^{(1)\gamma} C_{(ij)\gamma\beta}^\alpha, \quad \left(i = 0, 1, 2; j = 1, 2; \delta_{2\beta} = \dot{\partial}_{2\beta} \right)\end{aligned}$$

we find this formulae.

Proposition 4.2 The $(z^{(2)})$ -deflection fields are given by

$$\begin{aligned} D_i^{(2)\alpha} &= \frac{1}{2} \left(N_2^\alpha{}_\beta + M_2^\alpha{}_\beta \right) + \frac{1}{2} z^{(1)\gamma} \delta_{\beta 1} N_1^\alpha{}_\gamma + z^{(2)\gamma} L_{(i0)}^\alpha{}_{\gamma\beta}, \\ d_i^{(21)\alpha} &= -\frac{1}{2} \left(2N_2^\alpha{}_\beta - N_1^\alpha{}_\beta \right) + \frac{1}{2} z^{(1)\gamma} B_{11}^\alpha{}_{\gamma\beta} + z^{(2)\gamma} C_{(i1)}^\alpha{}_{\gamma\beta}, \\ d_i^{(22)\alpha} &= \delta_\beta^\alpha + \frac{1}{2} z^{(1)\gamma} B_{12}^\alpha{}_{\gamma\beta} + z^{(2)\gamma} C_{(i2)}^\alpha{}_{\gamma\beta}. \end{aligned}$$

5 The Ricci identities

Let $\check{D}\Gamma(\check{N}) = \left(\check{L}_{(i0)}^a{}_{b\delta}, \check{C}_{(i1)}^a{}_{b\delta}, \check{C}_{(i2)}^a{}_{b\delta} \right)$ the coupling of the metrical d-connection D , $D^\top\Gamma(\check{N}) = \left(L_{(i0)}^\alpha{}_{\beta\delta}, C_{(i1)}^\alpha{}_{\beta\delta}, C_{(i2)}^\alpha{}_{\beta\delta} \right)$ and $D^\perp\Gamma(\check{N}) = \left(L_{(i0)}^{\bar{\alpha}}{}_{\bar{\beta}\delta}, C_{(i1)}^{\bar{\alpha}}{}_{\bar{\beta}\delta}, C_{(i2)}^{\bar{\alpha}}{}_{\bar{\beta}\delta} \right)$ ($i = 0, 1, 2$) the induced tangent connection on $Osc^2\check{M}$ and the induced normal connection on $Osc^2\check{M}$ respectively.

Teoremă 5.1 For any d-vector fields X^α , the following Ricci identities hold:

$$\begin{aligned} X^\alpha|_{i\beta|i\gamma} - X^\alpha|_{i\beta|i\gamma} &= X^\delta R_{(0i)}^\delta{}_{\beta\gamma} - \overset{(i)}{T}_{(0)}^\sigma{}_{\beta\gamma} X^\alpha|_{i\sigma} - R_{(01)}^\sigma{}_{\beta\gamma} X^\alpha|_{i\sigma} - \\ &\quad - R_{(02)}^\sigma{}_{\beta\gamma} X^\alpha|_{i\sigma}, \\ X^\alpha|_{i\beta}|_{i\gamma} - X^\alpha|_{i\gamma}|_{i\beta} &= X^\delta P_{(1i)}^\delta{}_{\beta\gamma} - \overset{(i)}{C}_{(i1)}^\sigma{}_{\beta\gamma} X^\alpha|_{i\sigma} - \overset{(i)}{P}_{(11)}^\sigma{}_{\beta\gamma} X^\alpha|_{i\sigma} \\ &\quad - P_{(12)}^\sigma{}_{\beta\gamma} X^\alpha|_{i\sigma}, \\ X^\alpha|_{i\beta}|_{i\gamma} - X^\alpha|_{i\gamma}|_{i\beta} &= X^\delta P_{(2i)}^\delta{}_{\beta\gamma} - \overset{(i)}{C}_{(i2)}^\sigma{}_{\beta\gamma} X^\alpha|_{i\sigma} - \overset{(i)}{P}_{(21)}^\sigma{}_{\beta\gamma} X^\alpha|_{i\sigma} \\ &\quad - \overset{(i)}{P}_{(22)}^\sigma{}_{\beta\gamma} X^\alpha|_{i\sigma}, \end{aligned} \tag{5.3}$$

$$\begin{aligned}
X^\alpha \mid_{i\beta}^{(1)} \mid_{i\gamma}^{(2)} - X^\alpha \mid_{i\gamma}^{(2)} \mid_{i\beta}^{(1)} &= X^\delta Q_{(21)}^\delta{}^\alpha{}_{\beta\gamma} - C_{(i2)}^\sigma{}_{\beta\gamma} X^\alpha \mid_{i\sigma}^{(1)} - \\
&\quad - Q_{(22)}^\sigma{}_{\beta\gamma} X^\alpha \mid_{i\sigma}^{(2)},
\end{aligned}$$

$$\begin{aligned}
X^\alpha \mid_{i\beta}^{(j)} \mid_{i\gamma}^{(j)} - X^\alpha \mid_{i\gamma}^{(j)} \mid_{i\beta}^{(j)} &= X^\delta S_{(ji)}^\delta{}^\alpha{}_{\beta\gamma} - S_{(j)}^\sigma{}_{\beta\gamma} X^\alpha \mid_{i\sigma}^{(1)} - \\
&\quad - R_{(j2)}^\sigma{}_{\beta\gamma} X^\alpha \mid_{i\sigma}^{(2)}
\end{aligned}$$

unde $R_{(22)}^\alpha{}_{\beta\gamma} = 0, (i = 0, 1, 2, j = 1, 2)$.

The Ricci identities 5.3 applied to the Liouville d-vector fields $z^{(1)\alpha}$ and $z^{(2)\alpha}$ lead to the some fundamental identities.

Theorem 5.2 *The deflection tensor fields satisfy the following identities::*

$$\begin{aligned}
D_i^{(j)\alpha}{}_{\beta} \mid_{i\gamma} - D_i^{(j)\alpha}{}_{\gamma} \mid_{i\beta} &= z^{(j)\delta} R_{(0i)}^\delta{}^\alpha{}_{\beta\gamma} - D_i^{(j)}{}^\alpha{}_{\delta} T_{(0)}^\delta{}_{\beta\gamma} - \\
&\quad - d_i^{(j1)}{}^\alpha{}_{\delta} R_{(01)}^\delta{}_{\beta\gamma} - d_i^{(j2)}{}^\alpha{}_{\delta} R_{(02)}^\delta{}_{\beta\gamma}, \\
D_i^{(j)\alpha}{}_{\beta} \mid_{i\gamma} - d_i^{(j1)}{}^\alpha{}_{\gamma} \mid_{i\beta} &= z^{(j)\delta} P_{(1i)}^\delta{}^\alpha{}_{\beta\gamma} - D_i^{(j)}{}^\alpha{}_{\delta} C_{(i1)}^\delta{}_{\beta\gamma} - \\
&\quad - d_i^{(j1)}{}^\alpha{}_{\delta} P_{(11)}^\sigma{}_{\beta\gamma} - d_i^{(j2)}{}^\alpha{}_{\delta} P_{(12)}^\sigma{}_{\beta\gamma}, \\
D_i^{(j)\alpha}{}_{\beta} \mid_{i\gamma} - d_i^{(j2)}{}^\alpha{}_{\gamma} \mid_{i\beta} &= z^{(j)\delta} P_{(2i)}^\delta{}^\alpha{}_{\beta\gamma} - D_i^{(j)}{}^\alpha{}_{\delta} C_{(i2)}^\sigma{}_{\beta\gamma} - \\
&\quad - d_i^{(j1)}{}^\alpha{}_{\delta} P_{(21)}^\sigma{}_{\beta\gamma} - d_i^{(j2)}{}^\alpha{}_{\delta} P_{(22)}^\sigma{}_{\beta\gamma}, \\
d_i^{(j1)}{}^\alpha{}_{\beta} \mid_{i\gamma} - d_i^{(j2)}{}^\alpha{}_{\gamma} \mid_{i\beta} &= z^{(j)\delta} Q_{(2i)}^\delta{}^\alpha{}_{\beta\gamma} - X^\alpha \mid_{i\sigma}^{(1)} - \\
&\quad - d_i^{(j1)}{}^\alpha{}_{\delta} C_{(i2)}^\delta{}_{\beta\gamma} - d_i^{(j2)}{}^\alpha{}_{\delta} Q_{(22)}^\delta{}_{\beta\gamma},
\end{aligned} \tag{5.4}$$

$$\begin{aligned} d_{i\beta}^{(ji)\alpha} \big|_{i\gamma} - d_{i\gamma}^{(ji)\alpha} \big|_{i\beta} &= z^{(j)\delta} S_{(li)\delta}^{\alpha\beta\gamma} - \\ &- d_{i\delta}^{(jl)\alpha} S_{(i)\delta}^{\beta\gamma} - d_{i\delta}^{(j2)\alpha} R_{(l2)\delta}^{\beta\gamma}, \end{aligned}$$

$$\left(i = 0, 1, 2; j, l = 1, 2; R_{(22)\beta\gamma}^{\alpha} = 0. \right)$$

Also, if the $(z^{(1)})$ -and $(z^{(2)})$ -deflection tensors have the following particular form

$$\begin{aligned} D_{i\beta}^{(1)\alpha} &= 0, & d_{i\beta}^{(11)\alpha} &= \delta_{\beta}^{\alpha}, & d_{i\beta}^{(12)\alpha} &= 0 \\ D_{i\beta}^{(2)\alpha} &= 0, & d_{i\beta}^{(21)\alpha} &= 0, & d_{i\beta}^{(22)\alpha} &= \delta_{\beta}^{\alpha} \end{aligned} \quad (5.5)$$

then, the fundamental identities from 5.4 are very important, especially for applications.

Proposition 5.2 If the deflection tensor are given by 5.5, then the following identities hold:

$$\begin{aligned} z^{(j)\delta} R_{(0i)\delta}^{\alpha\beta\gamma} &= R_{(0j)\beta\gamma}^{\alpha}, & z^{(1)\delta} P_{(2i)\delta}^{\alpha\beta\gamma} &= P_{(21)\beta\gamma}^{\alpha}, & z^{(2)\delta} P_{(1i)\delta}^{\alpha\beta\gamma} &= P_{(12)\beta\gamma}^{\alpha}, \\ z^{(j)\delta} P_{(ji)\delta}^{\alpha\beta\gamma} &= P_{(jj)\beta\gamma}^{\alpha}, & z^{(1)\delta} Q_{(2i)\delta}^{\alpha\beta\gamma} &= C_{(i2)\beta\gamma}^{\alpha}, & z^{(2)\delta} Q_{(2i)\delta}^{\alpha\beta\gamma} &= \overset{i}{Q}_{(22)\beta\gamma}^{\alpha}, \\ z^{(j)\delta} S_{(ji)\delta}^{\alpha\beta\gamma} &= \overset{j}{S}_{(i)\beta\gamma}^{\alpha}, & z^{(1)\delta} S_{(2i)\delta}^{\alpha\beta\gamma} &= 0, & z^{(2)\delta} S_{(1i)\delta}^{\alpha\beta\gamma} &= R_{(12)\beta\gamma}^{\alpha}. \end{aligned} \quad (5.6)$$

$$(i = 0, 1, 2; j = 1, 2)$$

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References

- [1] Atanasiu Gh. *New aspects in differential geometry of the second order*, Seminarul de Mecanica, Sisteme Dinamice Diferentiale, Univ. de Vest din Timisoara, Facultatea de Matematica, vol. 82, 2001, 1-81
- [2] Atanasiu Gh., *Structure equations of second order*, Proc. of The 5-th Conference of Balkan Society of Geometers, August 29-September 2, 2005, Mangalia-Romania, BSG Proccedings 13, Geometry Balkan Press, Bucharest, Romania 2006, 18-27
- [3] Bejancu A., *On the theory of Finsler submanifolds*, in Finslerian Geometry, Kluwer Acad. Publ., Vol. 109, 2000, 111-129

- [4] Miron R., *The Geometry of higher order Lagrange spaces. Applications to mechanics and physics*, Kluwer Acad. Publ., FTPH no. 82, 1997
- [5] Miron R., Atanasiu Gh., *Higher order Lagrange spaces*, Rev. Roum. de Math. Pures et Appl. T. 41, No. 3-4, 1996, 251-263
- [6] Miron R., Atanasiu Gh., *Differential Geometry of the k -Osculator Bundle*, Rev. Roumaine, Math Pures et Appl., T 41, No. 3-4, 1996, 205-236
- [7] Miron R., Atanasiu Gh., *Lagrange Geometry of Second Order*, Math. Comput. Modelling, vol. 20, no. 4, 1994, 41-56
- [8] Miron R., Anastasiei M. *The Geometry of Lagrange Spaces: Theory and Applications*, Kluwer Acad. Publ., FTPH, no. 59, 1994
- [9] Miron R., Anastasiei M. *Vector bundles. Lagrange Spaces. Applications to Relativity*, Ed. Academiei Romane, 1987
- [10] Oana A., *The relative covariant derivate and induced connections in the theory of embeddings in the 2-osculator bundle*, The Riemannian Geometry and Applications, 21-25 iunie 2007, Brasov, Romania
- [11] Oana A., *The Ricci identities on submanifolds in the 2-osculator bundle*, National Seminar on Finsler, Lagrange and Hamilton Spaces, 21-25 septembrie 2006, Brasov, Romania